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Two Clifford's Theorems for Strongly Group-Graded Rings

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Let A be a strongly group-graded ring. In this paper we obtain the main results as following: (1) In A a similar result to Clifford's theorem of the decomposition of induced modules from normal subgroups, it corrects the result of [4]. (2) Frobenius Reciprocity for A. (3) We extend the theory of stable modules in two ways one of which is the localization theory of rings and modules. In some special cases we can obtain the corresponding results of [1, 2, 3].

INTRODUCTION

The decomposition of induced modules and characters is one of the most powerful methods for constructing irreducible representations of finite groups. For modules induced from normal subgroups, the main results are due to Frobenius, Clifford, and Gallagher. Considering group-graded Clifford systems, one of Clifford's theorems is obtained in a generalized form [1, Theorem 11.17] by Tucker, Conlon, and Ward. This result leads to the theory of projective representations. For strongly group-graded rings, Cline [2] extended the classical theory of projective representations and stable modules. For group-graded rings, Dade [3] strengthened the theory of stable modules of Cline. Also F. Van Oystaeyen [4], considering strongly group-graded rings, extended the classical Clifford's theorem of the decomposition of induced modules from normal subgroups.

Continuing this clue we also study strongly group-graded rings. This paper is composed of two parts. In Section 1 let the group be finite. First, we correct the result of F. Van Oystaeyen (Proposition 1.3); second, we strengthen another result of Cline (Theorem 1.8); finally, the critical fact of the theorem in [1] for modules induced from normal subgroups is described in language of module categories (Theorem 1.9). In Section 2 we give a general description of the theory of the stable modules by

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using the method of the localization of rings and modules (Theorem 2.5). Therefore we can strengthen the results corresponding to Cline and Dade (Corollary 2.7) and extend the classical Clifford's theorem [1, Theorem 11.17] (Corollary 2.8).

The last part convinces us the localization of rings and modules is one of powerful methods in representations of groups though it is well known that it is the most powerful technical methods in commutative algebras, general rings, and modules.

In this paper, we fix our notation, G will be a group with unit 1, A will always be a k-algebra with 1, over a commutative ring k with 1. We say that A is a G-graded k-algebra if $A=\bigoplus_{g\in G} A_g$ is a direct sum of k-submodules A_g, with $g\in G$ and $A_gA_h\subseteq A_{gh}$, for all $g,h\in G$. If $A_gA_h=A_{gh}$, for all g, $h\in G$, we say A is strongly G-graded. Further, if there exists some unit $a_g\in A$ with $A_g=a_gA_1=A_1a_g$, for all $g\in G$, we say A is G-graded Clifford system. It follows that $1_A\in A_1$, so that A_1 is a k-subalgebra with identity. For each $a\in A$, we denote by a_g the homogeneous component of a in A_g. Further details on graded rings may be found in [3, 7] and details on the methods of localization may be found in [12, 11] or [6, Chap. 7].

1. Finite Groups

Throughout this section G will be a finite group.

Let A be a strongly G-graded ring. Then $A_{g^{-1}}A_g=A_1$ for all $g\in G$. Thus we may fix a decomposition of 1,

$$1=\sum_{i=1}^{n_g} s_{g^{-1}}^{(i)}t_g^{(i)} \qquad (*)$$

with $s_{g^{-1}}^{(i)}\in A_{g^{-1}}$, $t_g^{(i)}\in A_g$, for each $g\in G$.

Lemma 1.1. *Let A be a strongly G-graded ring. Then A_{A_1} is a progenerator in* Mod-A_1, *which is the category of right unitary A_1-modules. Similar to ${}_{A_1}A$ in A_1-*Mod.

Proof. This result is a direct corollary of [2, Theorem 2.12]. But we can directly prove it by $(*)$ and the "dual basis lemma." ∎

Remark. Similarly, if G is infinite, then A_{A_1} $({}_{A_1}A)$ is projective as a right (left) A_1-module.

Lemma 1.2. *Let A be a strongly G-graded ring, W be a right A-module.*

Then the set

$$I = \{g \in G \mid W \otimes_{A_1} A_1 \simeq W \otimes_{A_1} A_g, \text{ as } A_1\text{-modules}\}$$

is a subgroup of G. I is called the inertia group of W in A, and we denote I by $\mathrm{Iner}_{A_1}(W)$, *or* $\mathrm{Iner}(W)$.

Proof. Let φ be an A_1-isomorphism,

$$W \otimes 1 \simeq W \otimes A_g, \qquad \text{where} \quad g \in G.$$

Define a map ψ from $W \otimes A_h$ to $W \otimes A_{gh}$ (for each $h \in G$) by

$$\psi(w \otimes a) = \varphi(w \otimes 1)\, a, \qquad \text{for all} \quad w \in W, a \in A_h.$$

We claim that ψ is an A_1-isomorphism. ψ is an A_1-epimorphism since

$$(W \otimes A_g)\, A_h = W \otimes A_g A_h = W \otimes A_{gh}, \qquad \text{for all} \quad g, h \in G.$$

ψ is monic since if $\psi(\sum_j w_j \otimes a_j) = 0$, i.e., $\sum_j \varphi(w_j \otimes 1)\, a_j = 0$, for some $w_j \in W$, $a_j \in A_h$. Then, by $(*)$,

$$\begin{aligned} \varphi\left(\sum_j w_j \otimes a_j s_{h^{-1}}^{(i)}\right) &= \sum_j \varphi(w_j \otimes 1)\, a_j s_{h^{-1}}^{(i)} \\ &= 0, \qquad \text{for all } i. \end{aligned}$$

By the definition of φ, $\sum_j w_j \otimes a_j s_{h^{-1}}^{(i)} = 0$, for all i. So for each i, $\sum_j w_j a_j s_{h^{-1}}^{(i)} = 0$ since there is a canonical A_1-isomorphism $W \otimes_{A_1} A_1 \simeq W$. Hence,

$$\begin{aligned} \sum_j w_j \otimes a_j &= \sum_j w_j \otimes a_j \sum_i s_{h^{-1}}^{(i)} t_h^{(i)} \\ &= \sum_{j,i} w_j \otimes a_j s_{h^{-1}}^{(i)} t_h^{(i)} = \sum_{j,i} w_j a_j s_{h^{-1}}^{(i)} \otimes t_h^{(i)} \\ &= \sum_i \left(\sum_j w_j a_j s_{h^{-1}}^{(i)}\right) \otimes t_h^{(i)} = 0. \end{aligned}$$

Thus ψ is an A_1-isomorphism, and we easily obtain the fact that I is a subgroup of G. ∎

Remark. When k is a field and A is a G-graded Clifford system k-algebra our Lemma 1.2 is [3, Proposition 11.15].

PROPOSITION 1.3. *Let A be a strongly G-graded ring. Consider a simple A-module $V \in$* Mod-*A. Then the following three statements hold:*

(i) *V is a f.g. semisimple right A_1-module, and if W is a simple A_1-submodule of V, then its simple summands are isomorphic to conjugates $\{W \otimes_{A_1} A_g \mid g \in G\}$ of W.*

(ii) *Let $A_I = \sum_{h \in I} A_h$, where $I = \mathrm{Iner}(W)$. Then A_I is a strongly I-graded ring, and there exists a natural number e such that $V \simeq e(\bigoplus_{g \in I \backslash G} W \otimes A_g)$ in* Mod-A_1. *And the homogeneous component U of A_1-module V containing W is a right A_I-module, and V is A-isomorphic to the induced module $U \otimes_{A_I} A$.*

(iii) *U is a simple A_I-submodule of $W \otimes_{A_1} A_I$, and* $\mathrm{Iner}_{A_I}(W) = I$.

Proof. We only give a outline of the proof. Let $U := \sum_{h \in I} WA_h = WA_I$, $U_g := \sum_{h \in I} WA_{hg} = WA_{Ig}$, where $A_I = \sum_{h \in I} A_h$, $A_{Ig} := \sum_{h \in I} A_{hg} = A_I A_g$, for all $g \in I \backslash G$, a right transversal to I in G. Then $U = \bigoplus_{i=1, h_i \in I}^{e} WA_{h_i} \simeq e(W \otimes 1)$ implies $U_g = \sum_{i=1}^{e} WA_{h_i g} = \bigoplus_{i=1}^{e} WA_{h_i g} \simeq e(W \otimes A_g)$, and $V = \bigoplus_{g \in I \backslash G} U_g$. Further, there exists an A_1-isomorphism φ_g from $U \otimes_{A_I} A_{Ig}$ into U_g, given by $\varphi_g(x \otimes a) = xa$ for all $x \in U$, $a \in A_{Ig}$. The rest of the proof refers to [3, Proposition 11.16, 4, Proposition 2.5]. ∎

Remark. In our Proposition 1.3, F. Van Oystaeyen [4, Proposition 2.5] claim that there is an A_1-isomorphism

$$V \simeq W \otimes_{A_1} A. \tag{F}$$

This is not true, for we can obtain the following

COUNTEREXAMPLE. Let G be a finite group with a presentation $\langle r, s \mid r^3 = 1 = s^2,\ srs = r \rangle$, and $V = \mathbb{C}$ as a vector space over the complex number field $\mathbb{C}$, $\mathrm{Aut}(V) = \mathbb{C}^* = \mathbb{C} - \{0\}$. Define a map φ from G to $\mathrm{Aut}(V)$ by

$$\varphi(s) = -1, \qquad \varphi(r^n) = 1, \qquad \text{for all } n.$$

Obviously, φ is a homomorphism between groups, so V can be regarded as a right $A = \mathbb{C}G$-module. By $\dim_{\mathbb{C}} V = 1$, V is a simple right A-module. Let $N = \langle r \rangle$, $A_1 = \mathbb{C}N$, $A_S = \mathbb{C}Ns$. Then N is a normal subgroup of G, $\mathbb{C}$-algebra $A = A_1 \oplus A_S$ is strongly $G/N \simeq \langle s \rangle$-graded, and V as a right A_1-module is simple. By above (F), we should have an A_1-isomorphism $V \simeq V \otimes_{A_1} A \simeq A \oplus V \otimes A_S$. Thus $V \simeq 2V$ as right $\mathbb{C}$-vector spaces, and $1 = \dim_{\mathbb{C}} V = \dim_{\mathbb{C}}(2V) = 2$, a contradiction.

LEMMA 1.4. *Let A be a strongly G-graded ring. For every right A_1-module W, there exists an A_1-isomorphism*

$$W \otimes_{A_1} A \simeq \mathrm{Hom}_{A_1}(A, W),$$

where $w\otimes a\mapsto \{w,a\}$ *and* $\{w,a\}: x\mapsto w(ax)_1$, *for all* $w\in W$, $a, x\in A$. *Similar to left* A_1*-module* W.

Proof. The proof refers to [6]. ▮

COROLLARY 1.5. *Let* A *be a strongly* G*-graded ring. Then*

(i) $A\simeq \operatorname{Hom}_{A_1}(A_{A_1}, A_1)$ *as* A_1*-*A*-bimodules.*

(ii) $A\simeq \operatorname{Hom}_{A_1}({}_{A_1}A, A_1)$ *as* A*-*A_1*-bimodules.* ▮

In order to prove the main results in this section we need the following:

PROPOSITION 1.6. (Frobenius Reciprocity). *Let* A *be a strongly* G*-graded ring,* W *be a right* A_1*-module,* V *be a right* A*-module. Then there are two natural isomorphisms of groups:*

(i) $Fr\colon \operatorname{Hom}_{A_1}(V, W)\simeq \operatorname{Hom}_A(V, W\otimes_{A_1} A)$ *given by* $Fr(\theta)\,v = \sum_{g\in G}\sum_{i=1}^{n_g}\theta(vs_{g^{-1}}^{(i)})\otimes t_g^{(i)}$.

(ii) $Fr'\colon \operatorname{Hom}_{A_1}(W, V)\simeq \operatorname{Hom}_A(W\otimes_{A_1} A, V)$ *given by* $Fr'(\beta)(w\otimes a) = (\beta w)\,a$.

Proof. We only prove (i). There are group-isomorphisms:

$$\begin{aligned}
&\operatorname{Hom}_A(V, W\otimes_{A_1} A)\\
&\quad\simeq \operatorname{Hom}_A(V, \operatorname{Hom}_{A_1}(A, W)) \qquad \text{(see Lemma 1.4)}\\
&\quad\simeq \operatorname{Hom}_{A_1}(V\otimes_A A, W)\\
&\quad\simeq \operatorname{Hom}_{A_1}(V, W). \qquad\qquad (1)
\end{aligned}$$

Thus the composite homomorphism of these isomorphisms is a group-isomorphism τ: $\operatorname{Hom}_A(V, W\otimes_{A_1} A)\simeq \operatorname{Hom}_{A_1}(V, W)$. We consider the behaviour of these isomorphisms in (1). Let $f\in \operatorname{Hom}_A(V, W\otimes_{A_1} A)$, $v\in V$, then $f(v)=\sum_{g\in G} w_g\otimes a_g = \sum_g w_g\otimes a_g\sum_{i=1}^{n_g} s_{g^{-1}}^{(i)}t_g^{(i)} = \sum_{g,i} w_g^{(i)}\otimes t_g^{(i)}$, where $w_g\in W$, $a_g\in A_g$, $w_g^{(i)} = w_g a_g s_{g^{-1}}^{(i)}$. By (1), f uniquely corresponds to $\tau(f) =: f'\in \operatorname{Hom}_{A_1}(V, W)$ with $f'(v) = \sum_{i=1}^{n_1} w_1^{(i)} = w_1^{(1)} = w_1 a_1$ (for $n_1 = 1$, $s_1^{(1)} = 1 = t_1^{(1)}$). In fact, $f' = \varphi p f$, where p is the canonical projective map from $W\otimes_{A_1} A$ into $W\otimes_{A_1} A_1$, φ is the canonical isomorphism from $W\otimes_{A_1} A_1$ to W. Moreover, $f = Fr(f')$ since

$$\begin{aligned}
f'(vs_{g^{-1}}^{(i)}) &= \varphi p f(vs_{g^{-1}}^{(i)}) = \varphi p(f(v)\,s_{g^{-1}}^{(i)})\\
&= \varphi p\left(\left(\sum_{h\in G}\sum_{j=1}^{n_h} w_h^{(j)}\otimes t_h^{(j)}\right)s_{g^{-1}}^{(i)}\right)\\
&= \sum_j w_g^{(j)} t_g^{(j)} s_{g^{-1}}^{(i)},
\end{aligned}$$

and

$$\begin{aligned} Fr(f')(v) &= \sum_{g \in G} \sum_{i=1}^{n_g} f'(vs_{g^{-1}}^{(i)}) \otimes t_g^{(i)} \\ &= \sum_{g,i,j} w_g^{(j)} t_g^{(j)} s_{g^{-1}}^{(i)} \otimes t_g^{(i)} \\ &= \sum_{g,j,i} w_g^{(j)} \otimes t_g^{(j)} s_{g^{-1}}^{(i)} t_g^{(i)} \\ &= \sum_{g,j} w_g^{(j)} \otimes t_g^{(j)} = f(v). \end{aligned}$$

Thus $f = Fr(f')$. By the uniqueness, for each $f' \in \mathrm{Hom}_{A_1}(V, W)$, there must exist $f \in \mathrm{Hom}_A(V, W \otimes_{A_1} A)$ such that $f' = \varphi pf$ and $Fr(f') = f$, i.e., $Fr(f') \in \mathrm{Hom}_A(V, W \otimes_{A_1} A)$ is independent on the decomposition of 1 in $A_{g^{-1}} A_g = A_1$ for all $g \in G$. ▮

PROPOSITION 1.7. *Let A be a strongly G-graded ring, W be a right A_1-module, and let E denote the endomorphism ring* $\mathrm{End}(V_A)$, *viewed as a ring of left operators on $V = W \otimes_{A_1} A$. For each $g \in G$, let*

$$E_g = \{ f \in E \mid f(W \otimes 1) \subseteq W \otimes A_g \}.$$

Then

(i) *For all $g, h \in G$, we have*

$$(W \otimes A_g) A_h \subseteq W \otimes A_{gh}; \; E_g(W \otimes A_h) \subseteq W \otimes A_{gh};$$

$$E_g E_h \subseteq E_{gh}, \qquad 1_E \in E_1, \qquad E = \bigoplus_{g \in E} E_g.$$

(ii) *Each element $\varphi \in \mathrm{Hom}_{A_1}(W \otimes 1, W \otimes A_g)$ extends to a unique element $\hat{\varphi} \in E_g$, given by $\hat{\varphi}(w \otimes a) := \varphi(w \otimes 1) a$ for all $w \in W$, $a \in A$. The map $\varphi \mapsto \hat{\varphi}$ defines an isomorphism of groups,*

$$\mathrm{Hom}_{A_1}(W \otimes 1, W \otimes A_g) \xrightarrow{\sim} E_g, \qquad \textit{for all} \quad g \in G,$$

and this defines an isomorphism of rings when $g = 1$.

(iii) *If* $\mathrm{Iner}_A(W)$ *is equal to G, then E is a G-graded Clifford system.*

Proof. Let $\mathrm{END}(V_A) = \sum_{g \in G} E_g$. By [7, Corollary I.2.11]

$$\mathrm{END}(V_A) = \bigoplus_{g \in G} E_g \qquad \text{(as additive groups)}$$

is a subring of E.

Part (iii) is obtained by [3, Corollary 5.14] since G is finite, so $\mathrm{END}(V_A) = E$ (see [3, Corollary 3.10]).

The rest of the proof refers to [1, Proposition 11.14]. ∎

Remark. If G is infinite, by [3, Proposition 4.8 and Corollary 5.14], then (i) and (ii) in Proposition 1.7 and (iii)′ hold, where

(iii)′ If $\mathrm{Iner}_A(W) = G$, then $\mathrm{END}(V_A)$ is a G-graded Clifford system.

The units of the G-graded Clifford system $\mathrm{END}(V_A)$ can be determined as following:

Assume $e_g \colon W \otimes 1 \simeq W \otimes A_g$ as A_1-modules, for each $g \in G$. By (ii), e_g extends to a unique element $\hat{e}_g \in E_g$. It is clear that $\hat{e}_g$ is an unit and $E_g = \hat{e}_g E_1 = E_1 \hat{e}_g$.

Throughout this paper units $\{\hat{e}_g \mid g \in G\}$ are as above.

Now we obtain the main result as following:

THEOREM 1.8. *Let A be a strongly G-graded ring, W be a right A_1-module, and let $E = \mathrm{End}(V_A)$, where $V = W \otimes_{A_1} A$. If $\mathrm{Iner}_A(W) = G$, then there exists an E-A-bimodule isomorphism*

$$E \otimes_{E_1} W \simeq W \otimes_{A_1} A.$$

where the action of A on the left tensor product is given by

$$(\gamma \otimes w)\, a := \gamma \hat{e}_g \otimes \hat{e}_g^{-1}(w \otimes a) \qquad \textit{for all} \quad a \in A_g,\, w \in W,\, \gamma \in E,$$

with E_1 and the unit $\hat{e}_g$ in E_g defined as in Remark of Proposition 1.7.

Proof. By [2, Theorem 3.4], there is an A-isomorphism

$$\psi \colon E \otimes_{E_1} W \simeq W \otimes_{A_1} A,$$
$$\hat{e}_g \otimes w \mapsto \hat{e}_g(w \otimes 1) \qquad \text{for all} \quad w \in W, g \in G.$$

We claim that ψ is E-homomorphic. For each $g \in G$, $w \in W, f \in E_1$, there exists $f' \in E_1$ such that $f\hat{e}_g = \hat{e}_g f'$. Hence

$$\begin{aligned}\psi(f(\hat{e}_g \otimes w)) &= \psi(f\hat{e}_g \otimes w) = \psi(\hat{e}_g f' \otimes w) = \psi(\hat{e}_g \otimes f'w)\\ &= \hat{e}_g(f'w \otimes 1) = \hat{e}_g f'(w \otimes 1)\\ &= f\hat{e}_g(w \otimes 1) = f(\psi(\hat{e}_g \otimes w)).\end{aligned}$$

Thus ψ is left E_1-homomorphic. And for each g, $h \in G$, $w \in W$, there exists $f \in E_1$ such that $\hat{e}_h \hat{e}_g = \hat{e}_{hg} f$. So

$$\begin{aligned}\psi(\hat{e}_h(\hat{e}_g \otimes w)) &= \psi(\hat{e}_h\hat{e}_g \otimes w) = \psi(\hat{e}_{hg} f \otimes w) = \psi(\hat{e}_{hg} \otimes fw) \\ &= \hat{e}_{hg}(fw \otimes 1) = \hat{e}_{hg} f(w \otimes 1) \\ &= \hat{e}_h\hat{e}_g(w \otimes 1) = \hat{e}_h(\psi(\hat{e}_g \otimes w).\end{aligned}$$

Therefore, ψ is E-homomorphic, E-A-isomorphic. ∎

Remark. If G is infinite then there exists an $\mathrm{END}(V_A)$-A-isomorphism ψ from $\mathrm{END}(V_A) \otimes_{E_1} W$ to $W \otimes_{A_1} A$, given by

$$\psi(\hat{e}_g \otimes w) = \hat{e}_g(w \otimes 1), \qquad \text{for all} \quad w \in W, g \in G.$$

THEOREM 1.9. *Let A be a strongly G-graded Artin ring, W be a simple right A_1-module with* $\mathrm{Iner}_A(W) = G$, *and let V be the tensor product $W \otimes_{A_1} A$, E be the endomorphism ring* $\mathrm{End}(V_A)$. *Then the following three statements hold*:

(i) *V_A is a balanced A-module, and ${}_E V$ is a progenerator in E-*Mod, *so ($E' = \mathrm{End}({}_E V)$, ${}_{E'} V_E^* = \mathrm{Hom}(V, E)$, ${}_E V_{E'}$, E) is a Morita context and E is Morita equivalent to E'.*

(ii) *By* (i), *there is an isomorphism θ from the lattice of right ideals I of E into the lattice of A-submodules U of V, given by*

$$\theta(I) = IV \simeq I \otimes_{E_1} W, \qquad \theta^{-1}(U) = \{\gamma \in E \mid \gamma V \subseteq U\}.$$

(iii) *The lattice isomorphism θ is functorial, in the sense that E-homomorphisms $f: I \to I'$, between right ideals of E, correspond bijectively to A-homomorphisms $f \otimes 1$: $I \otimes_{E_1} W \to I' \otimes_{E_1} W$.*

Proof. (i) By [4, Proposition 1.2(3)], A_1 is Artin. By "Density Theorem," W_{A_1} is balanced, i.e., the map $A_1 \to \mathrm{End}({}_{E_1} W)$, given by $a \mapsto a_r$, $a_r: w \mapsto wa$ for all $a \in A_1$, $w \in W$, is an epimorphism of rings, and an A_1-A_1-bimodule homomorphism, and ${}_{E_1} W$ is a f.d. E_1-vector space. Hence, there are A_1-A-homomorphisms,

$$\begin{aligned} A &\simeq A_1 \otimes_{A_1} A \twoheadrightarrow \mathrm{End}({}_{E_1} W) \otimes_{A_1} A \\ &\simeq \mathrm{Hom}_{E_1}(W, W \otimes_{A_1} A) \\ &\qquad \text{(for } {}_{A_1} A \text{ is a progenerator in } A_1\text{-Mod)} \\ &\simeq \mathrm{Hom}_E(E \otimes_{E_1} W, W \otimes_{A_1} A) \qquad \text{(Frobenius Reciprocity)} \\ &\simeq \mathrm{Hom}_E(W \otimes_{A_1} A, W \otimes_{A_1} A) = E'. \end{aligned}$$

Hence, the composite A_1-A-homomorphism of these homomorphisms is surjective. We note the behaviour of these homomorphisms, let $a \in A$, we have

$$\begin{aligned} a &\mapsto 1 \otimes a \mapsto 1_r \otimes a \\ &\mapsto a'(a'\colon w \mapsto w \otimes a \text{ for all } w \in W) \\ &\mapsto a''(a''\colon \theta \otimes w \mapsto \theta(w \otimes a) \text{ for all } \theta \in E,\, w \in W) \\ &\mapsto f_a \in E'. \end{aligned}$$

It is clear that, let $w \in W$, $x \in A_g$, if $w \otimes x = \hat{e}_g(w_g \otimes 1)$ for some $w_g \in W$, then $f_a(w \otimes x) = \hat{e}_g(w_g \otimes a) = (\hat{e}_g(w_g \otimes 1))\, a = (w \otimes x)\, a$, i.e., f_a is a_r which is the right translation defined by a. Hence, V_A is a balanced A-module.

Since ${}_{E_1}W$ is a f.d. E_1-vector space, it is a progenerator in E_1-Mod. It is clear that ${}_EV \simeq E \otimes_{E_1} W$ is a progenerator in E-Mod. Thus $(E, E' = \mathrm{End}({}_EV), {}_EV_{E'}, {}_{E'}V_E^* = \mathrm{Hom}({}_EV, E))$ is a Morita context and E is Morita equivalent to E'. Thus we complete the proof of (i).

(ii) By Morita theorem (see [6, p. 167]), there are E-E-isomorphism $\tau\colon V \otimes_{E'} V^* \simeq E$, given by $\tau(x \otimes y^*) = (x, y^*) = (x)\, y^*$ for all $x \in V$, $y^* \in V^*$; and E'-E'-isomorphism $\mu\colon V^* \otimes_E V \simeq E'$, given by $\mu(y^* \otimes x) = [y^*, x]$, $[y^*, x]\colon z \mapsto (z, y^*)\, x$ for all x, $z \in V$, $y^* \in V^*$, and an isomorphism θ from the lattice of right ideals I of E into the lattice of E'-submodules U of V, given by $\theta(I) = IV \simeq I \otimes_{E_1} W$ as right A-modules, and

$$\theta^{-1}(U) = \tau(U \otimes_{E_1} V^*) = (U, V^*).$$

U is an E'-submodule of V iff it is an A-submodule of V since V is balanced. $I := \theta^{-1}(U)$ and $I' := \{\gamma \in E \mid \gamma V \subseteq U\}$ are the same since $IV = (U, V^*)\, V = U[V^*, V] = UE' = U$, implies $I \subseteq I'$. But let $\gamma \in I'$, i.e., $\gamma V \subseteq U$, then $\gamma E = \gamma(V, V^*) = (\gamma V, V^*) \subseteq (U, V^*) = I$, this implies $\gamma \in I$ and $I' \subseteq I$. Hence $\theta^{-1}(U) = \{\gamma \in E \mid \gamma V \subseteq U\}$.

(iii) Since the functor $F = - \otimes_E V$ induces equivalence from Mod-E to Mod-E'. Given I, I', two right ideals of E, there are natural group-isomorphisms:

$$\begin{aligned} \mathrm{Hom}_E(I, I') &\simeq \mathrm{Hom}_{E'}(FI, FI') = \mathrm{Hom}_{E'}(I \otimes_E V, I' \otimes_E V) \\ &\simeq \mathrm{Hom}_A(I \otimes_E (E \otimes_{E_1} W), I' \otimes_E (E \otimes_{E_1} W)) \\ &\simeq \mathrm{Hom}_A(I \otimes_{E_1} W, I' \otimes_{E_1} W). \end{aligned}$$

It is clear that the map

$$\mathrm{Hom}_E(I, I') \ni f \mapsto f \otimes 1 \in \mathrm{Hom}(I \otimes_{E_1} W, I' \otimes_{E_1} W)$$

is bijective. ∎

Remarks. If A is a G-graded Clifford system k-algebra, over a field k. Then Theorem 1.9 implies [1, Theorem 11.17]. We strengthen the latter.

COROLLARY 1.10. *Let A be a strongly G-graded ring with A_1 a semisimple Artin ring, W be a simple right A_1-module with* $\mathrm{Iner}_A(W)=G$. *Denote* $W\otimes_{A_1} A$ *by* V, $\mathrm{End}_A(V)$ *by* E. *Then* $(E, A, {}_EV_A, {}_AV_E^*=\mathrm{Hom}({}_EV, E))$ *is a Morita context and A is Morita equivalent to E.*

2. INFINITE GROUPS

Throughout this section, G will be any group.

In Theorem 1.9, we have known that $\otimes_E V$: Mod-$E \to$ Mod-E' and $\mathrm{Hom}_{E'}(-, V)$: Mod-$E' \to$ Mod-E are inverse equivalences of categories.

Since V_A is balanced the category Mod-E' must be natural isomorphic to a full additive subcategory of Mod-A. Can the latter be described in a detailed way? Dade did not consider the category Mod-E' but obtained a beautiful result [3, Theorem 8.2] as following: Under the hypothesis of Theorem 1.9. We have $\mathrm{Hom}_A(-, V)$: $\mathrm{Mod}(A \mid W) \to$ Mod-E and $-\otimes_E V$: Mod-$E \to \mathrm{Mod}(A \mid W)$ are inverse equivalences of categories, where

$$\mathrm{Mod}(A \mid W)=\{X\in \text{Mod-}A \mid X \text{ is a } W\text{-primary } A_1\text{-module, i.e.,}$$
$$X \simeq W^{(I)} \text{ of copies of } W, \text{ in Mod-}A_1\}$$

is a full additive subcategory of Mod-A. Therefore, there exists a natural isomorphism

$$\text{Mod-}E' \simeq \mathrm{Mod}(A \mid W) \qquad (G \text{ is finite}).$$

It is a pity that for infinite group G the result of Dade cannot show more properties of category of modules, e.g., Theorem 1.9.

In fact, $\mathrm{Mod}(A \mid W)$ is a Grothendieck category with a projective generator V and the simple module W_{A_1} can be replaced by a semisimple module, or a Σ-quasiprojective module.

We shall show these results by the theory of localization of rings and modules. So we need to recall some concerned concepts and results. For convenience, we consider left modules in this section.

Let R be a ring, M be a left R-module. We say that an R-module N is M-generated if it is a quotient of a direct sum $M^{(I)}$ of copies of M. When every submodule of M is M-generated, M is called a self-generator. The full subcategory of R-Mod consisting of the submodules of M-generated modules will be denoted by $\sigma(M)$; it is a Grothendieck category [8].

We say that a module N is M-projective if, for every quotient module X of M, the canonical homomorphism

$$\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, X)$$

is an epimorphism and, in particular, M is quasiprojective when it is M-projective. If $M^{(I)}$ is quasiprojective for each set I, then we say that M is Σ-quasiprojective.

THEOREM 2.1 [8, Theorem 1.3]. *Let M be a Σ-quasiprojective module, $S = \mathrm{End}({}_R M)$ and J the two-sided ideal of S consisting of the endomorphisms which factor through a finitely generated submodule of M. Then J is an idempotent ideal of S and hence $\mathscr{F} = \{I \subset_S S \mid J \subset I\} = \{I \subset_S S \mid MI = M\}$ is a left Gabriel topology of S. Moreover, the following assertion holds*: $\mathrm{Hom}_{R_1}(M, -)$: $CD(M) \to (S, \mathscr{F})$-Mod *and* $M \otimes_S -$: $(S, \mathscr{F})$-Mod $\to CD[M]$ *are inverse equivalences of categories.*

Remark. Where $CD[M] = \{X \in R\text{-Mod} \mid \text{there exists an } R\text{-exact sequence: } M^{(I)} \to M^{(J)} \to X \to 0, \text{ for some sets } I, J\}$, the quotient category $(S, \mathscr{F})$-Mod associated with $\mathscr{F}$ is the full subcategory of S-Mod whose objects are the $\mathscr{F}$-closed modules.

By [8, Corollary 1.4], $CD[M]$, and $(S, \mathscr{F})$-Mod are Grothendieck categories with projective generators ${}_R M$ and ${}_S S$, respectively.

PROPOSITION 2.2 [8, Proposition 1.5]. *Let M be a Σ-quasiprojective module. If M is a self-generator then $CD[M] = \sigma[M]$.*

LEMMA 2.3. *Let M be a Σ-quasiprojective left R-module. Then the following hold for any set I*:

(i) *M is $M^{(I)}$-projective.*

(ii) *$M^{(I)}$ is a Σ-quasiprojective module.*

Proof. By [9, Proposition 16.10(2)], we have (i). By definition, (ii) holds. ∎

We now turn to consider stable modules of the strongly G-graded ring A.

LEMMA 2.4. *Let A be a strongly G-graded ring, W be a left A_1-module with $\mathrm{Iner}_A(W) = G$. Denote $A \otimes_{A_1} W$ by V, then the following assertions hold*:

(i) *W is Σ-quasiprojective, so is V;*

(ii) *W is a self-generator, so is V.*

In particular, when any of these conditions hold, $\sigma[V] = CD[V]$.

Proof. (i) By the hypothesis, $V = A \otimes_{A_1} W \simeq \bigoplus_{g \in G} A_g \otimes W \simeq W^{(G)}$ in A_1-Mod. Hence $V^{(I)} \simeq W^{(G \times I)}$ in A_1-Mod, for any set I. By Lemma 2.3, $W^{(I)}$ is $W^{(G \times I)}$-projective.

Let $V^{(I)} \to X \to 0$ by any A-exact sequence. Then it is also an A_1-exact sequence, and

$$\mathrm{Hom}_{A_1}(W^{(I)}, V^{(I)}) \to \mathrm{Hom}_{A_1}(W^{(I)}, X) \to 0$$

is an exact sequence. Frobenius Reciprocity gives a commutative diagram,

$$\begin{tikzcd} \mathrm{Hom}_A(V^{(I)}, V^{(I)}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_A(V^{(I)}, X) \arrow[d, "\wr"'] & \\ \mathrm{Hom}_{A_1}(W^{(I)}, V^{(I)}) \arrow[r] & \mathrm{Hom}_{A_1}(W^{(I)}, X) \arrow[r] & 0, \end{tikzcd}$$

where the homomorphism in the first row is the canonical homomorphism, by Snake lemma, it must be epic. Hence V is Σ-quasiprojective. We complete the proof of (i).

(ii) Let X be an A-submodule of V, p_g the projective map from $V = \bigoplus_{g \in G} A_g \otimes_{A_1} W$ to $W^g = A_1 \otimes_{A_1} W$ ($\simeq W$ in A_1-Mod). Then Xp_g is a A_1-submodule of W and there exists a set I_g and an A_1-exact sequence: $W^{(I_g)} \to Xp_g \to 0$. It follows that $\bigoplus_g W^{(I_g)} \to \bigoplus_g Xp_g \to \sum_g Xp_g = X \to 0$, and $\bigoplus_g W^{(I_g)} \to X \to 0$ are A_1-exact sequences. Since A_{A_1} is projective, also flat,

$$A \otimes_{A_1} \left(\bigoplus_g W^{(I_g)} \right) \to A \otimes_{A_1} X \to 0$$

and $V^{(\bigcup I_g)} \to AX = X \to 0$ are A-exact sequences, i.e., X is V-generated. Therefore V is a self-generator. ∎

Let A be a strongly G-graded ring, ${}_{A_1}W$ be a Σ-quasiprojective module. Let V be $A \otimes_{A_1} W$, E^* be $\mathrm{End}({}_A V)$, and E be $\mathrm{END}({}_A V) = \bigoplus_{g \in G} E_g$ (see Proof of Proposition 1.7) which is the graded endomorphism ring of the graded left A-module ${}_A V$. The following symbols are of Theorem 2.1. Since there is an isomorphism of rings $E_1 \simeq S = \mathrm{End}({}_{A_1} W)$ (see [3, Proposition 4.8]), we can identify E_1 with S, and set

$$(E|\mathscr{F})\text{-Mod} = \{X \in E\text{-Mod} \mid X \in (E_1, \mathscr{F})\text{-Mod}\}.$$

Then the following holds

THEOREM 2.5. *If* $\mathrm{Iner}_A(W) = G$, *then*

$$F = V \otimes_E -: (E|\mathscr{F})\text{-Mod} \to CD[V] \textit{ and } G = \mathrm{Hom}_A(V, -):$$

$CD[V] \to (E|\mathscr{F})$-Mod *are equivalences of categories.*

In particular, when ${}_{A_1}W$ *is finitely generated,* $\mathrm{Hom}_A(V,-)$ *induces an equivalence from* $CD[V]$ *to* E-Mod.

Proof. By Theorem 2.1,

$W\otimes_{E_1}-$: $(E_1,\mathscr{F})$-Mod $\to CD[W]$ and $\mathrm{Hom}_{A_1}(W,-)$: $CD[W]\to (E_1,\mathscr{F})$-Mod are inverse equivalences of categories.

In order to prove that F and G are inverse equivalences of categories we must show $F(X)=V\otimes_E X\in CD[V]$, for all $X\in(E|\mathscr{F})$-Mod, $G(Y)=\mathrm{Hom}_A(V,Y)\in(E|\mathscr{F})$-Mod, for all $Y\in CD[V]$, and $GF\approx 1_{(E|\mathscr{F})\text{-Mod}}$, $FG\approx 1_{CD[V]}$.

We prove these assertions one by one.

(1) For each $X\in(E|\mathscr{F})$-Mod, then $F(X)=V\otimes_E X\simeq(W\otimes_{E_1}E)\otimes_E X\simeq W\otimes_{E_1}X$, in A_1-Mod. But ${}_{E_1}X\in(E_1,\mathscr{F})$-Mod, so $W\otimes_{E_1}X\in CD[W]$, i.e., there exists an A_1-exact sequence

$$W^{(I)}\to W^{(J)}\to W\otimes_{E_1}X\to 0.$$

By Lemma 1.1, A_{A_1} is flat. Hence there exist A-exact sequences

$$A\otimes_{A_1}W^{(I)}\to A\otimes_{A_1}W^{(J)}\to A\otimes_{A_1}(W\otimes_{E_1}X)\to 0,$$

and

$$V^{(I)}\to V^{(J)}\to V\otimes_{E_1}X\to 0.$$

But we have a canonical A-exact sequence

$$V\otimes_{E_1}X\to V\otimes_E X\to 0.$$

Composing of these exact sequences, the sequence

$$V^{(I)}\to V^{(J)}\to V\otimes_E X\to 0$$

is exact. It follows $F(X)=V\otimes_E X\in CD[V]$.

(2) Let $Y\in CD[V]$, i.e., there exists an A-exact sequence

$$V^{(I)}\to V^{(J)}\to Y\to 0.$$

So there exists an A_1-exact sequence

$$W^{(G\times I)}\to W^{(G\times J)}\to Y\to 0.$$

It follows ${}_{A_1}Y\in CD[W]$. ${}_{E_1}G(Y)\in(E_1,\mathscr{F})$-Mod since $G(Y)=\mathrm{Hom}_A(V,Y)\simeq\mathrm{Hom}_{A_1}(W,Y)$ in E_1-Mod. Thus $G(Y)\in(E|\mathscr{F})$-Mod.

(3) For each $Y \in CD[V]$ there exists a commutative diagram,

$$\begin{array}{ccc} V \otimes_E \mathrm{Hom}_A(V, Y) & \xrightarrow{f} & Y \\ {\scriptstyle \xi}\wr\downarrow & & \| \\ W \otimes_{E_1} \mathrm{Hom}_{A_1}(W, Y) & \xrightarrow[\sim]{g} & Y \end{array}$$

where the A-homomorphism f are given by

$$f(v \otimes \theta) = (v)\,\theta, \qquad \text{for all} \quad v \in V, \theta \in \mathrm{Hom}_A(V, Y),$$

the A_1-isomorphism g are given by

$$g(w \otimes \theta_1) = (w)\,\theta_1, \qquad \text{for all} \quad w \in W, \theta_1 \in \mathrm{Hom}_{A_1}(W, Y),$$

and ξ is composed of the following natural E_1-isomorphisms:

$$\begin{aligned} V \otimes_E \mathrm{Hom}_A(V, Y) &\simeq (W \otimes_{E_1} E) \otimes_E \mathrm{Hom}_A(V, Y) \\ &\simeq W \otimes_{E_1} \mathrm{Hom}_A(A \otimes_{A_1} W, Y) \\ &\simeq W \otimes_{E_1} \mathrm{Hom}_{A_1}(W, \mathrm{Hom}_A(A, Y)) \\ &\simeq W \otimes_{E_1} \mathrm{Hom}_{A_1}(W, Y). \end{aligned}$$

Note: In these E_1-isomorphisms we have used the A-E-isomorphism (see the Remark of Theorem 1.8)

$$\psi\colon W \otimes_{E_1} E \simeq A \otimes_{A_1} W = V$$
$$w \otimes \hat{e}_g \mapsto (1 \otimes w)\,\hat{e}_g, \qquad \text{for all} \quad w \in W, g \in G.$$

In particular, $w \otimes 1_E \to 1_A \otimes w$, for all $w \in W$.

Let $a \in A_g$, $w \in W$, $\theta \in \mathrm{Hom}_A(V, Y)$. Then $v = a \otimes w = (1 \otimes w')\,\hat{e}_g$ for some $w' \in W$. And in these isomorphisms there are correspond relations as following:

$$\begin{aligned} v \otimes \theta = (1 \otimes w')\,\hat{e}_g \otimes \theta &\mapsto (w' \otimes \hat{e}_g) \otimes \theta \mapsto w' \otimes \hat{e}_g \theta \\ &\mapsto w' \otimes \alpha(a(w)\,\alpha := (a \otimes w)\,\hat{e}_g \theta, \text{ for all } a \in A_g, w \in W). \\ &\mapsto w' \otimes \beta((w)\,\beta := (1 \otimes w)\,\hat{e}_g \theta, \text{ for all } w \in W). \end{aligned}$$

Thus $\xi((a \otimes w) \otimes \theta) = w' \otimes \beta$, and $g\xi((a \otimes w) \otimes \theta) = g(w' \otimes \beta) = (w')\,\beta = (1 \otimes w')\,\hat{e}_g \theta = (a \otimes w)\,\theta = f((a \otimes w) \otimes \theta)$. Hence $g\xi = f$ since these elements $\{(a \otimes w) \otimes \theta\}$ generate $V \otimes_E \mathrm{Hom}_A(V, Y)$. But g is natural E_1-isomorphic, f is natural E_1-isomorphic, too. Therefore f is a natural E-isomorphism from $FG(Y)$ to Y, and $FG \approx 1_{CD[V]}$.

(4) For every $X \in (E|\mathscr{F})$-Mod, then $GF(X) = \mathrm{Hom}_A(V, V \otimes_E X)$ and there exists a commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{f_1} & \mathrm{Hom}_A(V, V \otimes_E X) \\ \| & & \wr \downarrow \eta \\ X & \xrightarrow[\sim]{g_1} & \mathrm{Hom}_{A_1}(W, W \otimes_{E_1} X), \end{array}$$

where the E-homomorphism f_1 is given by

$$(x) f_1 : v \mapsto v \otimes x, \qquad \text{for all} \quad x \in X, v \in V,$$

the E_1-isomorphism g_1 is given by

$$(x) g_1 : w \mapsto w \otimes x, \qquad \text{for all} \quad x \in X, w \in W,$$

and η is composed of following natural A_1-isomorphisms:

$$\begin{aligned} \mathrm{Hom}_A(V, V \otimes_E X) &= \mathrm{Hom}_A(A \otimes_{A_1} W, V \otimes_E X) \\ &\simeq \mathrm{Hom}_{A_1}(W, \mathrm{Hom}_A(A, V \otimes_E X)) \simeq \mathrm{Hom}_{A_1}(W, V \otimes_E X) \\ &\simeq \mathrm{Hom}_{A_1}(W, (W \otimes_{E_1} E) \otimes_E X) \simeq \mathrm{Hom}_{A_1}(W, W \otimes_{E_1} X). \end{aligned}$$

Let $a \in A_g$, $w \in W$, $\varphi \in \mathrm{Hom}_A(V, V \otimes_E X)$. Then in these isomorphisms there are corresponding relations,

$$\begin{aligned} &\varphi \mapsto \varphi_1((a(w))\, \varphi_1 = (a \otimes w)\, \varphi) \\ &\varphi_1 \mapsto \varphi_2((w)\, \varphi_1 = (1 \otimes w)\, \varphi) \\ &\varphi_2 \mapsto \varphi_3((w)\, \varphi_3 = \sum_i (w_i' \otimes \hat{e}_g) \otimes x_i) \\ &\varphi_3 \mapsto \varphi_4 = (\varphi)\, \eta \left((w)\, \varphi_4 = \sum_i w_i' \otimes \hat{e}_g x_i \right), \end{aligned}$$

where $(1 \otimes w)\, \varphi = \sum_i (a_i \otimes w_i) \otimes x_i$, for some $g \in G$, $a_i \in A_g$, and $a_i \otimes w_i = (1 \otimes w_i')\, \hat{e}_g$, $w_i \in W$. And let $\varphi(x) f_1$, for some $x \in X$, then

$$(1 \otimes w)(x) f_1 = (1 \otimes w) \otimes x \qquad \text{and} \qquad (1 \otimes w) = (1 \otimes w)\, 1_E.$$

Thus for each $w \in W$,

$$(w)((x) f_1 \eta) = w \otimes 1_E x = w \otimes x = (w)(x)\, g_1,$$

i.e.,

$$(x) f_1 \eta = (x)\, g_1, \qquad \text{for all} \quad x \in X, \text{ and } f_1 \eta = g_1.$$

Hence $f_1 = g_1 \eta^{-1}$ is a natural E_1-isomorphism, also a natural E-isomorphism from $GF(X)$ to X. It follows $GF \approx 1_{(E|\mathscr{F})\text{-Mod}}$. To sum up we know that (F, G) is an equivalence between the category $(E|\mathscr{F})$-Mod and the category $CD[V]$.

Finally, when ${}_{A_1}W$ is finitely generated. Then $\mathscr{F} = \{S\}$. Hence $(E|\mathscr{F})\text{-Mod} = E\text{-Mod}$, and (F, G) is an equivalence between the category E-Mod and the category $CD[V]$. ▮

COROLLARY 2.6. *Under the hypothesis of Theorem* 2.5, $(E|\mathscr{F})$-Mod *is a Grothendieck category with the projective generator* ${}_EE^* = \mathrm{End}({}_AV)$. *Further, let* J^* *be the two-sided ideal of* E^* *consisting of the endomorphisms which factor through a finitely generated submodule of* V. *Then* J^* *is an idempotent ideal of* E^* *and hence* $\mathscr{F}^* = \{I^* \subset {}_{E^*}E^* \mid J^* \subset I^*\} = \{I^* \subset {}_{E^*}E^* \mid VI^* = V\}$ *is a left Gabriel topology of* E^*. *Moreover, the following assertion holds*: $\mathrm{Hom}_A(V, -)$: $CD[V] \to (E^*, \mathscr{F}^*)$-Mod *and* $V \otimes_{E^*} -$: $(E^*, \mathscr{F}^*)\text{-Mod} \to CD[V]$ *are inverse equivalences of categories.*

Proof. By Lemma 2.4, ${}_AV$ is Σ-quasiprojective. So $CD[V]$ is a Grothendieck category with projective generator V, and $(E|\mathscr{F})$-Mod is a Grothendieck category with projective generator $\mathrm{Hom}_A(V, V) = E^*$ (see Theorem 2.5) and the last assertion holds (Theorem 2.1). ▮

Remark. (1) When ${}_{A_1}W$ is finitely generated $E = E^*$ [3, Corollary 3.10] and $(E|\mathscr{F})\text{-Mod} = E\text{-Mod}$.

(2) When G is finite $E = E^*$ [3, Corollary 3.10] $\mathrm{Hom}_A(V, -)$ induces equivalences from $CD[V]$ to $(E|\mathscr{F})$-Mod and $(E, \mathscr{F}^*)$-Mod.

COROLLARY 2.7. *Let* A *be a strongly* G*-graded ring,* W *be a semisimple left* A_1*-module. Then* W *is a* Σ*-quasiprojective module and a self-generator. Denote* $A \otimes_{A_1} W$ *by* V, $\mathrm{END}({}_AV)$ *by* E. *If* $\mathrm{Iner}_A(W) = G$, *then*

$$F = V \otimes_E -: (E|\mathscr{F})\text{-Mod} \to (A \mid W)\text{-Mod}$$

and

$$G = \mathrm{Hom}_A(V, -): (A \mid W)\text{-Mod} \to (E|\mathscr{F})\text{-Mod}$$

are inverse equivalences of categories, where the definition of $(E|\mathscr{F})$-Mod *as Theorem* 2.5, $(A \mid W)\text{-Mod} = \{Y \in A\text{-Mod} \mid Y$ *divides in* A_1-Mod *some direct sum of copies of* $W\}$. *When* W *is simple* $(A \mid W)$-Mod *is the same as the one given at the beginning of this section.*

Proof. By [10], the semisimple module W is Σ-quasiprojective, and, obviously, it is a self-generator. By Lemma 2.4, V is a self-generator in A-Mod, a Σ-quasiprojective module and $CD[V] = \sigma[V]$.

If $(A \mid W)$-Mod $= CD[V]$, then, by Theorem 2.5, the assertion holds. We only need to show this equality. For each $Y \in (A \mid W)$-Mod, i.e., $Y \mid W^{(I)}$, for some set I, there is an A_1-exact sequence $W^{(I)} \to W^{(I)} \to Y \to 0$.

Similar to the proof of Theorem 2.5, there exists an A-exact sequence $V^{(I)} \to V^{(I)} \to A \otimes_{A_1} Y \to Y \to 0$. Hence $Y \in CD[V]$ and $(A \mid W)$-Mod $\subseteq CD[V]$.

And for every $X \in CD[V]$, i.e., there is an A-exact sequence $V^{(I)} \to V^{(J)} \to X \to 0$. By $\mathrm{Iner}_A(W) = G$, $V = A \otimes_{A_1} W \simeq W^{(G)}$ as A_1-modules. $X \mid V^{(J)}$ $(\simeq W^{(J \times G)})$ in A_1-Mod since ${}_{A_1}V$ and ${}_{A_1}V^{(J)}$ are semisimples. Thus $X \in (A \mid W)$-Mod and $CD[V] \subseteq (A \mid W)$-Mod. Therefore $(A \mid W)$-Mod $= CD[V]$. ∎

Remark. In this corollary, Dade [3] told us that $(A \mid W)$-Mod is a additive full subcategory of A-Mod. But our result shows that $(A \mid W)$-Mod $= CD[V]$ is a Grothendieck category with a projective generator V.

COROLLARY 2.8. *Under the hypothesis of Theorem 2.5. If ${}_{A_1}W$ is a self-generator, in particular, it is semisimple, then the following assertions hold*:

(i) *There is an isomorphism θ from the lattice of the isomorphism classes of those left ideals I of E which are in* $(E \mid \mathscr{F})$-Mod *to the isomorphism classes of all A_1-modules U of V,*

$$\theta([I]) = [VI] = [W \otimes_{E_1} I],$$

$$\theta^{-1}([U]) = [\{\gamma \in E \mid V\gamma \subseteq U\}].$$

(ii) *There is a one-to-one correspondence between the isomorphism classes of those simple E-modules which are in* $(E \mid \mathscr{F})$-Mod *and the isomorphism classes of those simple A-modules which are in $\sigma[V]$.*

In particular, when ${}_{A_1}W$ is finitely generated the above θ is an isomorphism from the lattice of the isomorphism classes of left ideals of E to the lattice of the isomorphism classes of all A-submodules of V.

Proof. Part (ii) holds since $\mathrm{Hom}_A(V, -)$: $\sigma[V] \to (E \mid \mathscr{F})$-Mod and $V \otimes_E -$: $(E \mid \mathscr{F})$-Mod $\to \sigma[V]$ are inverse equivalences of categories.

Part (i) holds since ${}_AV$ is a self-generator hence every A-submodule of V is in $\sigma[V]$.

When ${}_{A_1}W$ is finitely generated $(E \mid \mathscr{F})$-Mod $= E$-Mod, it follows that the last assertion holds. ∎

Remark. This corollary is a general form of the classical Clifford's Theorem.

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